Stéphane PASQUET
http://www.bordeaux-cours-maths.fr
http://www.mathweb.fr

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Gaussian elimination method

Let the matrix :

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{pmatrix} = \left(a_{i,j}\right)$$

We have to find the reverse of the matrix A. For this, we will use the Gauss method. this method is to append the identity matrix to A :

$$\left(\begin{array}{ccc:ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

The objective is to obtain the identity matrix on the left.

Step 1 : we transform $a_{1,1}$ in « 1 »

For this, we divide the first line of A by « 2 » :

$$\left(\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} L_1 \leftarrow \frac{1}{2}L_1 \\ \\ \\ \end{array}$$

we indicate here the operation we perform on the line 1

Step 2 : we transform $a_{2,1}$ and $a_{3,1}$ in « 0 »

For this, we set the first line as pivot and, from line we want to transform, we add k times the pivot line.

$$\left(\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 2 & \frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - 2L_1 \end{array}$$

we add 1 times L_1 to L_2 to annulate $a_{2,1}$

we add -2 times L_1 to L_3 to annulate $a_{3,1}$

Step 3 : we transform $a_{2,2}$ in « 1 »

For this, we do like at « Step 1 » : we divide the second line by « 2 ».

$$\left(\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ L_2 \leftarrow \frac{1}{2}L_2 \\ \\ \end{array}$$

we divide all numbers by « 2 »

Step 4 : we transform $a_{2,3}$ in « 0 »

For this, we set the second line as pivot and, from the third line, we subtract the pivot line :

$$\left(\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{11}{4} & -\frac{5}{4} & -\frac{1}{2} & 1 \end{array}\right) \begin{array}{l} \\ \\ L_3 \leftarrow L_3 - L_2 \end{array}$$

Step 5 : we transform $a_{3,3}$ in « 1 »

For this, we divide the third line by $-\frac{11}{4}$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{11} & \frac{2}{11} & -\frac{4}{11} \end{array}\right) L_3 \leftarrow -\frac{4}{11}L_3$$

Step 6 : we transform $a_{3,2}$ and $a_{3,1}$ in « 0 »

For this, we set the third line as pivot and, from the lines we want to transform, we add k times the pivot line :

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{11} & -\frac{1}{11} & \frac{2}{11} \\ 0 & 1 & 0 & -\frac{1}{11} & \frac{4}{11} & \frac{3}{11} \\ 0 & 0 & 1 & \frac{5}{11} & \frac{2}{11} & -\frac{4}{11} \end{array}\right) \begin{array}{l} L_1 \leftarrow L_1 - \frac{1}{2}L_3 \\ L_2 \leftarrow L_2 - \frac{3}{4}L_3 \\ \end{array}$$

Here, $a_{2,1}$ is equal to « 0 » ; therefore, we don't have to change this number and we obtain the identity matrix on the left.
If this number was not equal to « 0 », we should have to take the second line like pivot and subtract at the first line $a_{2,1}$ times of the pivot line to obtain $a_{2,1} = 0$.

We obtained the reverse of the matrix A on the right :

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -1 & 2 \\ -1 & 4 & 3 \\ 5 & 2 & -4 \end{pmatrix}$$